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ON THE EXİSTENCE AND UNİQUENESS OF SOLUTİONS OF THE GOURSAT-DARBOUX SYSTEMS WİTH TWO-POİNT CONDİTİONS

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Abstract. The article investigates a system of second-order hyperbolic differential equations given with two-point boundary conditions. A Green's function for the boundary problems has been constructed, and the boundary problem has been transformed into an equivalent integral equation. Using the Banach contraction mapping principle, sufficient conditions for the existence and uniqueness of the solution have been found.

Keywords: Nonlocal problem, two-point boundary condition, Goursat-Darboux system, existence and uniqueness.

AMS Subject Classification: 35C05, 35L50, 35L70.

1. Introduction

The assessment of the current state of classical differential equations theory shows that non-local boundary problems hold a special place within the problems of mathematical physics. The emergence of such issues during the investigation of various problems in natural sciences and technology increases the interest in the study of non-local conditional problems. The study of non-local two-point boundary problems occupies an important place in the theory of non-local boundary problems. With such problems, it is usually not possible to directly measure the important characteristics of real processes, but the average value of those quantities is known. In such cases, when mathematically modeling these processes, this information can be represented in the form of a solution with multi-point boundary conditions.

It should be noted that non-local boundary problems arise in the construction of mathematical models of processes such as turbulence, plasma, heat transfer, demographic, and other processes [17,19,25,]. The article presented for the first time constructs a Green's function for a system of hyperbolic equations given with two-point boundary conditions and investigates the existence and uniqueness of the solution to the boundary problem.

2. The formulation of the problem and preliminary results.

In the paper we consider a nonlocal problem with two point boundary conditions for the Goursat-Darboux system un the domain $Q = [0,T] \times [0,l]$:

$$z_{tx} = f(t,x,z(t,x)),\ (t,x) \in Q, \tag{1}$$

$$z(0,x) + z(T,x) = \varphi(x),\ x \in [0,l], \tag{2}$$

$$z(t,0) + z(t,l) = \psi(t),\ t \in [0,T]. \tag{3}$$

where $z(t,x) = col(z_1(t,x), z(t,x), \dots, z_n(t,x))$ is an unknown n– dimensional vector-function, $f: Q \times R^n \to R^n$ is continuous on $Q \times R^n$ n - dimensional vector – functions $\varphi(x)$ and $\psi(t)$ are continuously – differentiable on $[0,T],[0,l]$ respectively.

It is assumed that the functionsand $\psi(t)$ satisfy the agreement condition

$$\varphi(0) + \varphi(l) = \psi(0) + \psi(T).$$

Note that problems for hyperbolic type equations have been studied in [1]-[8],[10]-[12],[18],[20],[24],[26],[27]. In these works the conditions of classical, general consistency of problems with nonlocal conditions have been established for second order hyperbolic equations. Similar issues for ordinary differential equations have been studied in [7], [9],[13]-[16] works.

3. Main results.

In this paper, for the first time the Green function is constructed for problem (1)-(3) and this problem is reduced to an equivalent integral equation. Further, using the method of Banach contraction mappings principle, sufficient conditions of classical consistency of the given problem are established.

Theorem 1. *A problem (1)-(3) is equivalent to the following integral equation:*

$$z(t,x) = \frac{1}{2}\psi(t) - \frac{1}{4}\varphi(l) - \frac{1}{4}\varphi(0) + \frac{1}{2}\varphi(x) + \int_0^T \int_0^l G(t,x,\tau,s) f(\tau,s,z) d\tau ds,$$

where

$$G(t,x,\tau,s) = \begin{cases} \frac{1}{4}, 0 \le \tau \le t, 0 \le s \le x, \\ -\frac{1}{4}, 0 \le \tau \le t, x < s \le l, \\ -\frac{1}{4}, t < \tau \le T, 0 \le s \le x, \\ \frac{1}{4}, t < \tau \le T, x < s \le l. \end{cases}$$

Proof. We will look for any solution of equation (1) in the form

$$z(t,x)=\int_0^t\int_0^x f(\tau,s,z(\tau,s)d\tau ds+a(t)+b(x), \tag{4}$$

here $a(t)$ and $b(x)$ are unknown continuous functions and are determined in the segments $[0,T]$, $[0,l]$ respectively. Let the function determined by equality (4) satisfy conditions (2) and (3). Then

$$z(0,x)=\int_0^0\int_0^x f(\tau,s,z(\tau,s))d\tau ds+a(0)+b(x),$$

$$z(T,x)=\int_0^T\int_0^x f(\tau,s,z(\tau,s))d\tau ds+a(T)+b(x),$$

$$z(0,x)+z(T,x)=\int_0^T\int_0^x f(\tau,s,z(\tau,s))d\tau ds+a(0)+a(T)+2b(x)=\varphi(x). \tag{5}$$

$$z(t,0)=\int_0^t\int_0^0 f(\tau,s,z(\tau,s))d\tau ds+a(t)+b(0),$$

$$z(t,l)=\int_0^t\int_0^l f(\tau,s,z(\tau,s))d\tau ds+a(t)+b(l),$$

$$z(t,0)+z(t,l)=\int_0^t\int_0^l f(\tau,s,z(\tau,s))d\tau ds+b(l)+b(o)+2a(t)=\psi(t). \tag{6}$$

Without loss of generality, we assume that the relationship

$$a(0)+a(T)=0$$

is valid.

From equality (5) we obtain the following relationship:

$$b(x)=\frac{1}{2}\varphi(x)-\frac{1}{2}\int_0^T\int_0^x f(\tau,s,z(\tau,s))d\tau ds,$$

$$b(l)=\frac{1}{2}\varphi(l)-\frac{1}{2}\int_0^T\int_0^l f(\tau,s,z(\tau,s))d\tau ds, \tag{7}$$

$$b(0)=\frac{1}{2}\varphi(0)-\frac{1}{2}\int_0^T\int_0^0 f(\tau,s,z(\tau,s))d\tau ds=\frac{1}{2}\varphi(0).$$

We'll take into account equality (6) the function $b(0), b(l)$ determined by equality (7).

Then

$$\int_0^t\int_0^l f(\tau,s,z(\tau,s))d\tau ds+\frac{1}{2}\varphi(l)-\frac{1}{2}\int_0^T\int_0^l f(\tau,s,z(\tau,s))d\tau ds+\frac{1}{2}\varphi(0)+2a(t)=\psi(t),$$

$$a(t)=\frac{1}{2}\psi(t)-\frac{1}{4}\varphi(l)-\frac{1}{4}\varphi(0)+\frac{1}{4}\int_0^T\int_0^l f(\tau,s,z(\tau,s))d\tau ds-\frac{1}{2}\int_0^t\int_0^l f(\tau,s,z(\tau,s))d\tau ds. \tag{8}$$

We’ll take into account expressions (7) and (8) obtained for the functions $a(t)$ and $b(x)$ in the (4).
Then

$$z(t,x)=\frac{1}{2}\psi(t)-\frac{1}{4}\varphi(l)-\frac{1}{4}\varphi(0)+\frac{1}{2}\varphi(x)+$$
$$+\frac{1}{4}\int_0^T\int_0^l f(\tau,s,z(\tau,s))d\tau ds-\frac{1}{2}\int_0^t\int_0^l f(\tau,s,z(\tau,s))d\tau ds-$$
$$-\frac{1}{2}\int_0^T\int_0^x f(\tau,s,z(\tau,s))d\tau ds+\int_0^t\int_0^x f(\tau,s,z(\tau,s))d\tau ds,\ (t,x)\in Q. \tag{9}$$

From the equality (9), we obtain that

$$z(t,x)=\frac{1}{2}\psi(t)-\frac{1}{4}\varphi(l)-\frac{1}{4}\varphi(0)+\frac{1}{2}\varphi(x)+$$
$$+\frac{1}{4}\int_0^t\int_0^x f\left(\tau,s,z(\tau,s)\right)d\tau ds-\frac{1}{4}\int_0^t\int_x^l f\left(\tau,s,z(\tau,s)\right)d\tau ds-$$
$$-\frac{1}{4}\int_t^T\int_0^x f\left(\tau,s,z(\tau,s)\right)d\tau ds+\frac{1}{4}\int_t^T\int_x^l f\left(\tau,s,z(\tau,s)\right)d\tau ds. \tag{10}$$

Let us introduce a new matrix function.

$$G(t,x,\tau,s)=\begin{cases}\dfrac{1}{4},0\le\tau\le t,0\le s\le x,\\ -\dfrac{1}{4},0\le\tau\le t,x\le s\le l,\\ -\dfrac{1}{4},t\le\tau\le T,0\le s\le x,\\ \dfrac{1}{4},t\le\tau\le T,x\le s\le l.\end{cases}$$

Then the equality (11) can be rewritten in the following form.

$$z(t,x)=\frac{1}{2}\psi(t)-\frac{1}{4}\varphi(l)-\frac{1}{4}\varphi(0)+\frac{1}{2}\varphi(x)+\int_0^T\int_0^l G(t,x,\tau,s)f(\tau,s,z)d\tau ds$$

which confirms the validity of (10). Let the function $z(t,x)$ be expressed by equality (10). We will show that the function $z(t,x)$ is the solution of (1), (2). We will calculate the derivative of $z(t,x)$ with respect to the variables t and x.

$$z_{tx}(t,x)=\frac{\partial^2}{\partial t\partial x}\left[\frac{1}{2}\psi(t)-\frac{1}{4}\varphi(l)-\frac{1}{4}\varphi(0)+\frac{1}{2}\varphi(x)\right]+$$
$$+\frac{\partial^2}{\partial t\partial x}\left[\frac{1}{4}\int_0^t\int_0^x f(\tau,s,z(\tau,s))d\tau ds\right]-\frac{\partial^2}{\partial t\partial x}\left[\frac{1}{4}\int_0^t\int_x^l f(\tau,s,z(\tau,s))d\tau ds\right]-$$
$$-\frac{\partial^2}{\partial t\partial x}\left[\frac{1}{4}\int_t^T\int_0^x f(\tau,s,z(\tau,s))d\tau ds\right]+\frac{\partial^2}{\partial t\partial x}\left[\frac{1}{4}\int_t^T\int_x^l f(\tau,s,z(\tau,s))d\tau ds\right]=$$
$$=\left(\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}\right)f(t,x,z(t,x))=f(t,x,z(t,x)).$$

Now we'll prove the second part of the theorem. We show that the function determined by equality (11) satisfies conditions (2) and (3).

$$z(0,x)+z(T,x)=\frac{1}{2}\psi(0)-\frac{1}{4}\varphi(l)-\frac{1}{4}\varphi(0)+\frac{1}{2}\varphi(x)+$$
$$+\frac{1}{4}\int_0^T\int_0^l f(\tau,s,z(\tau,s))d\tau ds-\frac{1}{2}\int_0^0\int_0^l f(\tau,s,z(\tau,s))d\tau ds-$$
$$-\frac{1}{2}\int_0^T\int_0^x f(\tau,s,z(\tau,s))d\tau ds+\int_o^0\int_0^x f(\tau,s,z(\tau,s))d\tau ds+$$
$$+\frac{1}{2}\psi(T)-\frac{1}{4}\varphi(l)-\frac{1}{4}\varphi(0)+\frac{1}{2}\varphi(x)+$$
$$+\frac{1}{4}\int_0^T\int_0^l f(\tau,s,z(\tau,s))d\tau ds-\frac{1}{2}\int_0^T\int_0^l f(\tau,s,z(\tau,s))d\tau ds-$$
$$-\frac{1}{2}\int_0^T\int_0^x f(\tau,s,z(\tau,s))d\tau ds+\int_0^T\int_0^x f(\tau,s,z(\tau,s))d\tau ds=$$
$$=\frac{1}{2}\psi(0)+\frac{1}{2}\psi(T)-\frac{1}{2}\varphi(l)-\frac{1}{2}\varphi(0)+\varphi(x)+$$
$$+\frac{1}{2}\int_0^T\int_0^l f(\tau,s,z(\tau,s))d\tau ds-\int_0^T\int_0^x f(\tau,s,z(\tau,s))d\tau ds-$$

$$-\frac{1}{2}\int_0^T\int_0^l f\left(\tau,s,z(\tau,s)\right)d\tau ds+\int_0^T\int_0^x f\left(\tau,s,z(\tau,s)\right)d\tau ds=$$

$$=\frac{1}{2}\left(\psi(0)+\psi(T)\right)-\frac{1}{2}\left(\varphi(l)+\varphi(0)\right)+\varphi(x)=\varphi(x).$$

We can show that the condition

$$z(t,0)+z(t,l)=\psi(t)$$

is also satisfied similarly.

This completed the proof of Theorem 1.

4. Existence and uniqueness.

To prove the uniqueness of the solution of the stated problem, we determined the operator $P:C\left(Q;R^n\right)\to C\left(Q;R^n\right)$ as

$$(Pz)(t,x)=\frac{1}{2}\psi(t)-\frac{1}{4}\varphi(l)-\frac{1}{4}\varphi(0)+\frac{1}{2}\varphi(x)+\int_0^T\int_0^l G(t,x,\tau,s)\,f(\tau,s,z)d\tau ds.$$

It is known that problem (1)-(3) is equivalent to the problem on a fixed point $z=Pz.$ So, problem (1)-(3) has a solution if and only if the operator P has a fixed point.

Theorem 2. *Assume that the following conditions:*

$$\left|f(t,x,z_2)-f\left(t,x,z_1\right)\right|\le M\left|z_2-z_1\right| \tag{11}$$

are satisfied for each $\left(t,x\right)\in Q$ *and for all* $z_1,z_2\in R^n$, *the constant* $M\ge 0$ *and*

$$L=lTSM<1, \tag{12}$$

where

$$S=\max_{Q\times Q}\left\|G(t,x,\tau,s\right\|.$$

Then boundary value problem (1)-(3) has a unique solution on Q.

Proof. Denoting

$$N=\max_{Q}\left|\frac{1}{2}\psi(t)-\frac{1}{4}\varphi(l)-\frac{1}{4}\varphi(0)+\frac{1}{2}\varphi(x)\right|,$$

$$\max_{(t,x)\in Q}\left|f(t,x,0)\right|=M_f,$$

and choose $r\ge\dfrac{N+M_fTS}{1-L}$. We'll prove that $PB_r\subset B_r$, where $B_r=\left\{x\in C\left(Q,R^n\right):\left\|z\right\|\le r\right\}$.

For $z\in B_r$ we have

$$\|Pz(t,x)\| \le N + \int_0^T\int_0^l |G(t,x,\tau,s)|\left(\left| f(\tau,s,z(\tau,s)) - f(\tau,s,0\right| + \left| f(\tau,s,0)\right|\right)d\tau ds \le$$

$$\le N + S\int_0^T\int_0^l \left(M|z| + M_f\right)dtdx \le \|N\| + SMrTl + M_f TlS \le \frac{\|N\| + M_f TS}{1-L} \le r.$$

Further, by (12), for any $z_1, z_z \in B_r$

$$|Pz_2 - Pz_1| \le \int_0^T\int_0^l |G(t,x,\tau,s)|\left| f(\tau,s,z_2(\tau,s)) - f(\tau,s,z_z(\tau,s)\right|d\tau ds \le$$

$$\le S\int_0^T\int_0^l M|z_2(t,x) - z_1(t,x)|dtdx \le MSTl \max_Q |z_2(t,x) - z_1(t,x)| \le$$

$$MSTl\|z_2 - z_1\|$$

is valid, or

$$\|Pz_2 - Pz_1\| \le L\|z_2 - z_1\|.$$

It is clear that by condition (12) P is contraction operator. This, boundary value problem (1)-(3) has a unique solution.

5. Example.

We give an example illustrating the main result obtained in the paper. Let's consider the following system of differential equations with an two-point boundary condition:

$$\begin{cases} y_{1tx} = \cos(0{,}1 y_2), \\ y_{2tx} = \dfrac{|y_1|}{(9+e^{tx})(1+|y_1|}, \end{cases} \quad (t,x) \in [0,1]\times[0,1], \tag{13}$$

$$\begin{cases} z_1(0,x) + z_1(1,x) = x, z_2(0,x) + z_2(1,x) = x^2, \\ z_1(t,0) + z_1(t,1) = t, z_2(t,0) + z_2(t,1) = t^2. \end{cases} \tag{14}$$

Obviously, the agreement condition is satisfied. Condition (13) is satisfied due to conditions (12) and $G_{\max} \le 1, M = 0.1.$ Consequently,

$$L = G_{\max} MTl = 1\cdot 0{,}1 = 0{,}1 < 1.$$

So, by theorem 2, boundary value problem (14)-(15) has a unique solution on $[0,1]\times[0,1].$

6. Conclusion.

In this paper, the existence and uniqueness of solutions for nonlinear hyperbolic differential equations with two-point boundary conditions is established. Note that the method introduced in the paper can be successfully used in more complicated problems for hyperbolic differential equation. For example, we can consider the following problem:

$$z_{tx} = f(t, x, z(t, x)), \qquad (t, x) \in Q,$$

with two point boundary condition

$$A_1 z(0, x) + A_2 z(T, x) = \varphi(x), \qquad x \in [0, l],$$
$$B_1 z(t, 0) + B_2 z(t, l) = \psi(t), \qquad t \in [0, T].$$

here $A_1, A_2, B_1, B_2 \in R^{n \times n}$ are the given matrices and $\det(A_1 + A_2) \neq 0, \det(B_1 + B_2) \neq 0$. $\varphi(x), x \in [0, l], \psi(t), t \in [0, T]$ are the given differentiated functions.

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